

# Backward Stochastic Differential Equations Driven by $G$ -Brownian Motion

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## Abstract

In this paper, we study the following of backward stochastic differential equations driven by a  $G$ -Brownian motion  $(B_t)_{t \geq 0}$  in the following form:

$$\begin{aligned} Y_t = & \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) d\langle B \rangle_s \\ & - \int_t^T Z_s dB_s - (K_T - K_t). \end{aligned}$$

Under a Lipschitz condition of  $f$  and  $g$  in  $Y$  and  $Z$ . The existence and uniqueness of the solution  $(Y, Z, K)$  is proved, where  $K$  is a decreasing  $G$ -martingale.

**Key words:**  $G$ -expectation,  $G$ -Brownian motion,  $G$ -martingale, Backward SDEs

**MSC-classification:** 60H10, 60H30

## 1 Introduction

A typical classical Backward Stochastic Differential Equation, BSDE in short, is defined on a Wiener probability space  $(\Omega, \mathcal{F}, P)$  in which  $\Omega$  is the space of continuous paths. A standard Brownian motion is defined as the canonical process, namely  $B_t(\omega) = \omega_t$ , for  $\omega \in \Omega$ , together with its natural filtration

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$\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ . The problem is to solve a pair of  $\mathbb{F}$ -adapted processes  $(Y, Z)$  satisfying the following BSDE

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad (1.1)$$

where  $g$  is a given function, called the generator of (1.1), and  $\xi$  is a given  $\mathcal{F}_T$ -measurable random variable called the terminal condition of the BSDE.

Linear BSDE was introduced by Bismut [2, 1973]. The basic existence and uniqueness theorem of nonlinear BSDEs, with a Lipschitz condition of  $g$  with respect to  $(y, z)$ , was obtained in Pardoux & Peng [10, 1990]. Peng [11, 1991a] established a probabilistic interpretation, through BSDE, of system of quasilinear partial differential equations, PDE in short, of parabolic and elliptic types, under a strong elliptic assumption. Then Peng [13, 1992] and Pardoux & Peng [12, 1992] obtained this interpretation for possibly degenerate situation. This interpretation which established a 1-1 correspondence between a solution of a PDE and the corresponding state dependent BSDE is the so-called nonlinear Feynman-Kac formula. Since then and specially after the study of BSDE in [6] with application to finance, BSDE theory has been extensively studied. We refer to a survey paper of [23] for more details of the theoretical studies and applications to, e.g., stochastic controls, optimizations, games and finance.

Under some suitable condition imposed to the generator  $g$ , this BSDE was used to define a nonlinear expectation  $\mathcal{E}^g[\xi] := Y_0$ , called  $g$ -expectation (see [14, Peng1997]). This  $g$ -expectation is time consistent, namely the conditional expectation  $\mathcal{E}^g[\xi|\mathcal{F}_t]$  is well-defined, under which the solution process  $Y_t$  is a nonlinear martingale  $Y_t = \mathcal{E}^g[\xi|\mathcal{F}_t]$ . In fact it was proved that there exists a 1-1 correspondence between a set of ‘dominated’ and time-consistent nonlinear expectations and that of BSDEs (see [3]).

There are at least two reasons to study BSDEs and/or the corresponding time-consistent nonlinear expectations outside of a classical probability space framework. The first one is that the classical BSDE can provide a probabilistic interpretation of a PDE only for quasilinear but not fully nonlinear cases. The second one is that the well-known HJB-equation method of volatility model uncertainty (see [1]) is difficult to treat a general path-dependent situation to measure financial risks. This problem is also closely related to defining an important type of time-consistent coherent risk measures (or sublinear expectation) for which the probabilities involved in the robust representation theorem are singular from each others.

The notion of time-consistent fully nonlinear expectations has been established in [15, Peng2004] and [16, Peng2005]. The main approach of [15] is to establish a new type of ‘path-dependent value function’ of a stochastic optimal control system, in which the time consistency was able to be obtained through the corresponding path-dependent dynamic programming principle (DPP).

In [16] a canonical space of nonlinear Markovian paths was defined. A nonlinear expectation, together with its time-consistent conditional expectations, was defined firstly on a subspace of finite-dimensional cyclic functions of canonical paths, through a sublinear Markovian semigroup, step by step and backwardly in time. This expectation and the corresponding conditional expectations were

then extended to the completion of the above subspace of cyclic functions by using the Banach norm induced by this sublinear expectation. Existence and uniqueness of a type of multi-dimensional fully nonlinear BSDE was obtained in this paper.

As a typical and important situation of the above nonlinear Markovian processes, Peng (2006) introduced a framework of time consistent nonlinear expectation called  $G$ -expectation  $\hat{\mathbb{E}}[\cdot]$  (see lecture notes of [22] and the references therein) in which a new type of Brownian motion called  $G$ -Brownian motion was constructed and the corresponding stochastic calculus of Itô's type was established.

Using this stochastic calculus the existence and uniqueness of SDEs driven by  $G$ -Brownian motion can be obtained, in a way parallel to that of classical theory of SDE, through which a large set of fully nonlinear Markovian and non Markovian processes can be easily generated. But the corresponding BSDE driven by a  $G$ -Brownian motion  $(B_t)_{t \geq 0}$  becomes a challenging and fascinating problem.

Just like in the classical situation, the first and most simplest BSDE in this  $G$ -framework is the corresponding  $G$ -martingale representation theorem. For a dense family of  $G$ -martingales, Peng [18] obtained the following result: a  $G$ -martingale  $M$  is of the form

$$\begin{aligned} M_t &= M_0 + \bar{M}_t + K_t, \\ \bar{M}_t &:= \int_0^t z_s B_s, \quad K_t := \int_0^t \eta_s \langle B \rangle_s - \int_0^t 2G(\eta_s) ds. \end{aligned}$$

Here  $M$  is decomposed into two types of very different  $G$ -martingales: the first one  $\bar{M}$  is called symmetric  $G$ -martingale for which  $-\bar{M}$  is also a  $G$ -martingale. The second one  $K$  is quite unusual since it is a decreasing process. How to understand this new type of decreasing  $G$ -martingales has become a main concern in the theory of  $G$ -framework, which rised an interesting open problem (see [18] and [22]).

An important step is to decompose an  $G$ -martingale  $M$  into a sum of a symmetric  $G$ -martingale  $\bar{M}$  and a decreasing  $G$ -martingale  $K$ . This difficult problem was solved after a series of successive efforts of Soner, Touzi & Zhang [25, 2011] and Song [27, 2011], [28, 2012]. Another important step is to give a completion of random variables in which the non increasing  $G$ -martingales  $K$  in the decomposition of the  $G$ -martingale  $\hat{\mathbb{E}}_t[\xi]$  can be uniquely represented  $K_t := \int_0^t \eta_s \langle B \rangle_s - \int_0^t 2G(\eta_s) ds$ . Thanks to an original new norm introduced in Song [28, 2012] for decreasing  $G$ -martingales, a representation theorem of  $G$ -martingales in a complete subspace of  $L_G^\alpha(\Omega_T)$  has been obtained by Peng, Song and Zhang [24, 2012].

In considering the above  $G$ -martingale representation theorem, a natural formulation of a BSDE driven by  $G$ -Brownian motion is to find a triple of

processes  $(Y, Z, K)$ , where  $K$  is a decreasing  $G$ -martingale, satisfying

$$\begin{aligned} Y_t = & \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) d\langle B \rangle_s \\ & - \int_t^T Z_s dB_s - (K_T - K_t). \end{aligned} \quad (1.2)$$

The main result of this paper is the existence and uniqueness of a triple  $(Y, Z, K)$  which solves BSDE (1.2), see Theorem 4.1 and 4.2.

To prove the existence and uniqueness, two new approaches have been introduced. The first one is applying the partition of unity theorem to construct a new type of Galerkin approximation, in the place of the well-known Picard approximation approach frequently used in classical BSDE theory. The second one involves Lemma 3.4 for decreasing  $G$ -martingales, which helps us to use our  $G$ -stochastic calculus obtain the uniqueness, as well as the existence part of the proof. Estimate (2.1) originally obtained in [27] also plays an important role.

Now let us compare the results of this paper with the existing results concerning fully nonlinear BSDEs.

For the case where the generator  $f$  in (1.2) is independent of  $z$  and  $g = 0$ , the above problem can be equivalently formulated as

$$Y_t = \hat{\mathbb{E}}_t[\xi + \int_t^T f(s, Y_s) ds].$$

The existence and uniqueness of such fully nonlinear BSDE was obtained in [16, Peng2005] and [18], [22]. This approach was used to treat many interesting problem corresponding fully nonlinear PDE and/or system of fully nonlinear PDEs, in which each component  $u^i$  of the solution is associated to its own second order nonlinear elliptic operator (see [22]). But a drawback of this formulation is that it is difficult to treat the case where the generators  $f$  and/or  $g$  contain the  $z$ -terms (but the  $z$ -term can be integrated in the nonlinear Markovian semigroup, see [16, Peng2005] and [22]).

Soner, Touzi and Zhang [26, 2012] have obtained an existence and uniqueness theorem for a type of fully nonlinear BSDE, called 2BSDE, whose generator can contain  $Z$ -term. Their solution is  $(Y, Z, K^\mathbb{P})_{\mathbb{P} \in \mathcal{P}_H^\kappa}$  which solves, for each probability  $\mathbb{P} \in \mathcal{P}_H^\kappa$ , the following BSDE

$$Y_t = \xi + \int_t^T F_s(Y_s, Z_s) ds - \int_t^T Z_s dB_s + (K_T^\mathbb{P} - K_t^\mathbb{P}), \quad \mathbb{P}\text{-a.s.},$$

for which the following minimum condition is satisfied

$$K_t^\mathbb{P} = \text{ess} \inf_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[K_T^\mathbb{P}], \quad \mathbb{P}\text{-a.s.}, \quad \forall \mathbb{P} \in \mathcal{P}_H^\kappa, \quad t \in [0, T].$$

But in their paper the processes  $(K^\mathbb{P})_{\mathbb{P} \in \mathcal{P}_H^\kappa}$  are not able to be ‘‘aggregated’’ into an ‘universal  $K$ ’. This is a drawback in the sense that the quantity of calculation for solving this 2BSDE is still involved an complicated optimization problem with respect to the original subset of probabilities  $\mathcal{P}_H^\kappa$ . In our paper the

triple  $(Y, Z, K)$  is universally defined within the  $G$ -Brownian motion framework. The method of our paper can be also applied to many other situations.

The paper is organized as follows. In section 2, we present some preliminaries for stochastic calculus under  $G$ -framework. Some estimates for the solution of  $G$ -BSDE are established in section 3. In section 4 the existence and uniqueness theory is provided.

## 2 Preliminaries

We review some basic notions and results of  $G$ -expectation and the related space of random variables. More details of this section can be found in [17], [18], [19], [20], [22].

**Definition 2.1** *Let  $\Omega$  be a given set and let  $\mathcal{H}$  be a vector lattice of real valued functions defined on  $\Omega$ , namely  $c \in \mathcal{H}$  for each constant  $c$  and  $|X| \in \mathcal{H}$  if  $X \in \mathcal{H}$ .  $\mathcal{H}$  is considered as the space of random variables. A sublinear expectation  $\hat{\mathbb{E}}$  on  $\mathcal{H}$  is a functional  $\hat{\mathbb{E}} : \mathcal{H} \rightarrow \mathbb{R}$  satisfying the following properties: for all  $X, Y \in \mathcal{H}$ , we have*

- (a) *Monotonicity: If  $X \geq Y$  then  $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$ ;*
- (b) *Constant preservation:  $\hat{\mathbb{E}}[c] = c$ ;*
- (c) *Sub-additivity:  $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$ ;*
- (d) *Positive homogeneity:  $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$  for each  $\lambda \geq 0$ .*

$(\Omega, \mathcal{H}, \hat{\mathbb{E}})$  is called a sublinear expectation space.

**Definition 2.2** *Let  $X_1$  and  $X_2$  be two  $n$ -dimensional random vectors defined respectively in sublinear expectation spaces  $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$  and  $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$ . They are called identically distributed, denoted by  $X_1 \stackrel{d}{=} X_2$ , if  $\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)]$ , for all  $\varphi \in C_{l.Lip}(\mathbb{R}^n)$ , where  $C_{l.Lip}(\mathbb{R}^n)$  is the space of real continuous functions defined on  $\mathbb{R}^n$  such that*

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y| \quad \text{for all } x, y \in \mathbb{R}^n,$$

where  $k$  and  $C$  depend only on  $\varphi$ .

**Definition 2.3** *In a sublinear expectation space  $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ , a random vector  $Y = (Y_1, \dots, Y_n)$ ,  $Y_i \in \mathcal{H}$ , is said to be independent of another random vector  $X = (X_1, \dots, X_m)$ ,  $X_i \in \mathcal{H}$  under  $\hat{\mathbb{E}}[\cdot]$ , denoted by  $Y \perp X$ , if for every test function  $\varphi \in C_{l.Lip}(\mathbb{R}^m \times \mathbb{R}^n)$  we have  $\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}]$ .*

**Definition 2.4** ( $G$ -normal distribution) *A  $d$ -dimensional random vector  $X = (X_1, \dots, X_d)$  in a sublinear expectation space  $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$  is called  $G$ -normally distributed if for each  $a, b \geq 0$  we have*

$$aX + b\bar{X} \stackrel{d}{=} \sqrt{a^2 + b^2}X,$$

where  $\bar{X}$  is an independent copy of  $X$ , i.e.,  $\bar{X} \stackrel{d}{=} X$  and  $\bar{X} \perp X$ . Here the letter  $G$  denotes the function

$$G(A) := \frac{1}{2} \hat{\mathbb{E}}[\langle AX, X \rangle] : \mathbb{S}_d \rightarrow \mathbb{R},$$

where  $\mathbb{S}_d$  denotes the collection of  $d \times d$  symmetric matrices.

Peng [20] showed that  $X = (X_1, \dots, X_d)$  is  $G$ -normally distributed if and only if for each  $\varphi \in C_{l.Lip}(\mathbb{R}^d)$ ,  $u(t, x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)]$ ,  $(t, x) \in [0, \infty) \times \mathbb{R}^d$ , is the solution of the following  $G$ -heat equation:

$$\partial_t u - G(D_x^2 u) = 0, \quad u(0, x) = \varphi(x).$$

The function  $G(\cdot) : \mathbb{S}_d \rightarrow \mathbb{R}$  is a monotonic, sublinear mapping on  $\mathbb{S}_d$  and  $G(A) = \frac{1}{2} \hat{\mathbb{E}}[\langle AX, X \rangle] \leq \frac{1}{2} |A| \hat{\mathbb{E}}[|X|^2] =: \frac{1}{2} |A| \bar{\sigma}^2$  implies that there exists a bounded, convex and closed subset  $\Gamma \subset \mathbb{S}_d^+$  such that

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}[\gamma A],$$

where  $\mathbb{S}_d^+$  denotes the collection of nonnegative elements in  $\mathbb{S}_d$ .

In this paper we only consider non-degenerate  $G$ -normal distribution, i.e., there exists some  $\underline{\sigma}^2 > 0$  such that  $G(A) - G(B) \geq \underline{\sigma}^2 \text{tr}[A - B]$  for any  $A \geq B$ .

**Definition 2.5** *i) Let  $\Omega_T = C_0([0, T]; \mathbb{R}^d)$ , the space of real valued continuous functions on  $[0, T]$  with  $\omega_0 = 0$ , be endowed with the supremum norm and let  $B_t(\omega) = \omega_t$  be the canonical process. Set  $\mathcal{H}_T^0 := \{\varphi(B_{t_1}, \dots, B_{t_n}) : n \geq 1, t_1, \dots, t_n \in [0, T], \varphi \in C_{l.Lip}(\mathbb{R}^{d \times n})\}$ .  $G$ -expectation is a sublinear expectation defined by*

$$\hat{\mathbb{E}}[X] = \tilde{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0} \xi_1, \dots, \sqrt{t_m - t_{m-1}} \xi_m)],$$

*for all  $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}})$ , where  $\xi_1, \dots, \xi_m$  are identically distributed  $d$ -dimensional  $G$ -normally distributed random vectors in a sublinear expectation space  $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$  such that  $\xi_{i+1}$  is independent of  $(\xi_1, \dots, \xi_i)$  for every  $i = 1, \dots, m-1$ .  $(\Omega_T, \mathcal{H}_T^0, \hat{\mathbb{E}})$  is called a  $G$ -expectation space.*

*ii) Let us define the conditional  $G$ -expectation  $\hat{\mathbb{E}}_t$  of  $\xi \in \mathcal{H}_T^0$  knowing  $\mathcal{H}_t^0$ , for  $t \in [0, T]$ . Without loss of generality we can assume that  $\xi$  has the representation  $\xi = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}})$  with  $t = t_i$ , for some  $1 \leq i \leq m$ , and we put*

$$\begin{aligned} \hat{\mathbb{E}}_{t_i}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}})] \\ = \tilde{\varphi}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_i} - B_{t_{i-1}}), \end{aligned}$$

where

$$\tilde{\varphi}(x_1, \dots, x_i) = \hat{\mathbb{E}}[\varphi(x_1, \dots, x_i, B_{t_{i+1}} - B_{t_i}, \dots, B_{t_m} - B_{t_{m-1}})].$$

Define  $\|\xi\|_{p,G} = (\hat{\mathbb{E}}[|\xi|^p])^{1/p}$  for  $\xi \in \mathcal{H}_T^0$  and  $p \geq 1$ . Then for all  $t \in [0, T]$ ,  $\hat{\mathbb{E}}_t[\cdot]$  is a continuous mapping on  $\mathcal{H}_T^0$  w.r.t. the norm  $\|\cdot\|_{1,G}$ . Therefore it can be extended continuously to the completion  $L_G^1(\Omega_T)$  of  $\mathcal{H}_T^0$  under the norm  $\|\cdot\|_{1,G}$ .

Let  $L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \dots, B_{t_n}) : n \geq 1, t_1, \dots, t_n \in [0, T], \varphi \in C_{b.Lip}(\mathbb{R}^{d \times n})\}$ , where  $C_{b.Lip}(\mathbb{R}^{d \times n})$  denotes the set of bounded Lipschitz functions on  $\mathbb{R}^{d \times n}$ . Denis et al. [5] proved that the completions of  $C_b(\Omega_T)$  (the set of bounded continuous function on  $\Omega_T$ ),  $\mathcal{H}_T^0$  and  $L_{ip}(\Omega_T)$  under  $\|\cdot\|_{p,G}$  are the same and we denote them by  $L_G^p(\Omega_T)$ .

**Definition 2.6** Let  $M_G^0(0, T)$  be the collection of processes in the following form: for a given partition  $\{t_0, \dots, t_N\} = \pi_T$  of  $[0, T]$ ,

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j, t_{j+1})}(t),$$

where  $\xi_i \in L_{ip}(\Omega_{t_i})$ ,  $i = 0, 1, 2, \dots, N-1$ . For  $p \geq 1$  and  $\eta \in M_G^0(0, T)$ , let  $\|\eta\|_{H_G^p} = \{\hat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}]\}^{1/p}$ ,  $\|\eta\|_{M_G^p} = \{\hat{\mathbb{E}}[\int_0^T |\eta_s|^p ds]\}^{1/p}$  and denote by  $H_G^p(0, T)$ ,  $M_G^p(0, T)$  the completions of  $M_G^0(0, T)$  under the norms  $\|\cdot\|_{H_G^p}$ ,  $\|\cdot\|_{M_G^p}$  respectively.

**Theorem 2.7** ([5, 7]) There exists a tight subset  $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$ , the set of probability measures on  $(\Omega_T, \mathcal{B}(\Omega_T))$ , such that

$$\hat{\mathbb{E}}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi] \text{ for all } \xi \in \mathcal{H}_T^0.$$

$\mathcal{P}$  is called a set that represents  $\hat{\mathbb{E}}$ .

**Remark 2.8** Denis et al. [5] gave a concrete set  $\mathcal{P}_M$  that represents  $\hat{\mathbb{E}}$ . For simplicity, we only introduce the 1-dimensional case, i.e.,  $\Omega_T = C_0([0, T]; \mathbb{R})$ .

Let  $(\Omega^0, \mathcal{F}^0, P^0)$  be a probability space and  $\{W_t\}$  be a 1-dimensional Brownian motion under  $P^0$ . Let  $F^0 = \{\mathcal{F}_t^0\}$  be the augmented filtration generated by  $W$ . Denis et al. [5] proved that

$$\mathcal{P}_M := \{P_h : P_h = P^0 \circ X^{-1}, X_t = \int_0^t h_s dW_s, h \in L_{F^0}^2([0, T]; [\underline{\sigma}, \bar{\sigma}])\}$$

is a set that represents  $\hat{\mathbb{E}}$ , where  $L_{F^0}^2([0, T]; [\underline{\sigma}, \bar{\sigma}])$  is the collection of  $F^0$ -adapted measurable processes with  $\underline{\sigma} \leq |h_s| \leq \bar{\sigma}$ . Here

$$\underline{\sigma}^2 := -\hat{\mathbb{E}}[-B_1^2] \leq \hat{\mathbb{E}}[B_1^2] =: \bar{\sigma}^2.$$

For this 1-dimensional case,

$$G(a) = \frac{1}{2} \hat{\mathbb{E}}[aB_1^2] = \frac{1}{2} [\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-].$$

Let  $\mathcal{P}$  be a weakly compact set that represents  $\hat{\mathbb{E}}$ . For this  $\mathcal{P}$ , we define capacity

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega_T).$$

A set  $A \subset \Omega_T$  is polar if  $c(A) = 0$ . A property holds “quasi-surely” (q.s. for short) if it holds outside a polar set. In the following, we do not distinguish two random variables  $X$  and  $Y$  if  $X = Y$  q.s.. We set

$$\mathbb{L}^p(\Omega_t) := \{X \in \mathcal{B}(\Omega_t) : \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty\} \text{ for } p \geq 1.$$

It is important to note that  $L_G^p(\Omega_t) \subset \mathbb{L}^p(\Omega_t)$ . We extend  $G$ -expectation  $\hat{\mathbb{E}}$  to  $\mathbb{L}^p(\Omega_t)$  and still denote it by  $\hat{\mathbb{E}}$ , for each  $X \in \mathbb{L}^1(\Omega_T)$ , we set

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X].$$

For  $p \geq 1$ ,  $\mathbb{L}^p(\Omega_t)$  is a Banach space under the norm  $(\hat{\mathbb{E}}[|\cdot|^p])^{1/p}$ .

Furthermore, we extend the definition of conditional  $G$ -expectation. For each fixed  $t \geq 0$ , let  $(A_i)_{i=1}^n$  be a partition of  $\mathcal{B}(\Omega_t)$ , and set

$$\xi = \sum_{i=1}^n \eta_i I_{A_i},$$

where  $\eta_i \in L_G^1(\Omega_T)$ ,  $i = 1, \dots, n$ . We define the corresponding conditional  $G$ -expectation, still denoted by  $\hat{\mathbb{E}}_s[\cdot]$ , by setting

$$\hat{\mathbb{E}}_s[\sum_{i=1}^n \eta_i I_{A_i}] := \sum_{i=1}^n \hat{\mathbb{E}}_s[\eta_i] I_{A_i} \text{ for } s \geq t.$$

The following lemma shows that the above definition of conditional  $G$ -expectation is meaningful.

**Lemma 2.9** *For each  $\xi, \eta \in L_G^1(\Omega_T)$  and  $A \in \mathcal{B}(\Omega_t)$ , if  $\xi I_A \geq \eta I_A$  q.s., then  $\hat{\mathbb{E}}_t[\xi] I_A \geq \hat{\mathbb{E}}_t[\eta] I_A$  q.s..*

**Proof.** Otherwise, we can choose a compact set  $K \subset A$  with  $c(K) > 0$  such that  $(\hat{\mathbb{E}}_t[\xi] - \hat{\mathbb{E}}_t[\eta])^- > 0$  on  $K$ . Since  $K$  is compact, we can choose a sequence of nonnegative functions  $\{\zeta_n\}_{n=1}^\infty \subset C_b(\Omega_t)$  such that  $\zeta_n \downarrow I_K$ . By Theorem 31 in [5], we have

$$\hat{\mathbb{E}}[\zeta_n(\xi - \eta)^-] \downarrow \hat{\mathbb{E}}[I_K(\xi - \eta)^-]$$

and

$$\hat{\mathbb{E}}[\zeta_n \hat{\mathbb{E}}_t[(\xi - \eta)^-]] \downarrow \hat{\mathbb{E}}[I_K \hat{\mathbb{E}}_t[(\xi - \eta)^-]].$$

Since

$$\hat{\mathbb{E}}[\zeta_n(\xi - \eta)^-] = \hat{\mathbb{E}}[\zeta_n \hat{\mathbb{E}}_t[(\xi - \eta)^-]],$$

we have

$$\hat{\mathbb{E}}[I_K \hat{\mathbb{E}}_t[(\xi - \eta)^-]] = \hat{\mathbb{E}}[I_K(\xi - \eta)^-] = 0.$$

Noting that

$$(\hat{\mathbb{E}}_t[\xi] - \hat{\mathbb{E}}_t[\eta])^- \leq \hat{\mathbb{E}}_t[(\xi - \eta)^-],$$

we get  $\hat{\mathbb{E}}_t[(\xi - \eta)^-] > 0$  on  $K$ . Also by  $c(K) > 0$  we get  $\hat{\mathbb{E}}[I_K \hat{\mathbb{E}}_t[(\xi - \eta)^-]] > 0$ . This is a contradiction and the proof is complete.  $\square$



We set

$$\mathbb{L}_G^{0,p,t}(\Omega_T) := \{\xi = \sum_{i=1}^n \eta_i I_{A_i} : A_i \in \mathcal{B}(\Omega_t), \eta_i \in L_G^p(\Omega), n \in \mathbb{N}\}.$$

We have the following properties.

**Proposition 2.10** *For each  $\xi, \eta \in \mathbb{L}_G^{0,1,t}(\Omega_T)$ , we have*

- (i) *Monotonicity: If  $\xi \leq \eta$ , then  $\hat{\mathbb{E}}_s[\xi] \leq \hat{\mathbb{E}}_s[\eta]$  for any  $s \geq t$ ;*
- (ii) *Constant preserving: If  $\xi \in \mathbb{L}_G^{1,t}(\Omega_t)$ , then  $\hat{\mathbb{E}}_t[\xi] = \xi$ ;*
- (iii) *Sub-additivity:  $\hat{\mathbb{E}}_t[\xi + \eta] \leq \hat{\mathbb{E}}_t[\xi] + \hat{\mathbb{E}}_t[\eta]$ ;*
- (iv) *Positive homogeneity: If  $\xi \in \mathbb{L}_G^{0,\infty,t}(\Omega_t)$  and  $\xi \geq 0$ , then  $\hat{\mathbb{E}}_t[\xi\eta] = \xi\hat{\mathbb{E}}_t[\eta]$ ;*
- (v) *Consistency: For  $t \leq s \leq r$ , we have  $\hat{\mathbb{E}}_s[\hat{\mathbb{E}}_r[\xi]] = \hat{\mathbb{E}}_s[\xi]$ .*
- (vi)  $\hat{\mathbb{E}}[\hat{\mathbb{E}}_t[\xi]] = \hat{\mathbb{E}}[\xi]$ .

**Proof.** (i) is direct consequence of Lemma 2.9. (ii)-(v) are obvious from the definition. We only prove  $\hat{\mathbb{E}}[\hat{\mathbb{E}}_t[\xi]] = \hat{\mathbb{E}}[\xi]$  for  $\xi$  which is bounded and positive.

Step 1. For  $\xi = \sum_{i=1}^N I_{K_i} \eta_i$ , where  $K_i$ ,  $i = 1, \dots, N$ , are disjoint compact sets and  $\eta_i \geq 0$ , we can choose  $\varphi_m^i \in C_b(\Omega_t)$  such that  $\varphi_m^i \downarrow K_i$  and  $\varphi_m^i \varphi_m^j = 0$  for  $i \neq j$ . By the same analysis as that in Lemma 2.9, we can get  $\hat{\mathbb{E}}[\sum_{i=1}^N I_{K_i} \hat{\mathbb{E}}_t[\eta_i]] = \hat{\mathbb{E}}[\sum_{i=1}^N I_{K_i} \eta_i]$ .

Step 2. For  $\xi = \sum_{i=1}^N I_{A_i} \eta_i$ , where  $A_i$ ,  $i = 1, \dots, N$ , are disjoint sets and  $\eta_i \geq 0$ . For each fixed  $P \in \mathcal{P}$ , we can choose compact sets  $K_m^i$  such that  $K_m^i \uparrow$  and  $P(A_i - K_m^i) \downarrow 0$ , then

$$\begin{aligned} E_P[\sum_{i=1}^N I_{A_i} \hat{\mathbb{E}}_t[\eta_i]] &= \lim_{m \rightarrow \infty} E_P[\sum_{i=1}^N I_{K_m^i} \hat{\mathbb{E}}_t[\eta_i]] \\ &\leq \lim_{m \rightarrow \infty} \hat{\mathbb{E}}[\sum_{i=1}^N I_{K_m^i} \hat{\mathbb{E}}_t[\eta_i]] \\ &= \lim_{m \rightarrow \infty} \hat{\mathbb{E}}[\sum_{i=1}^N I_{K_m^i} \eta_i] \\ &\leq \hat{\mathbb{E}}[\sum_{i=1}^N I_{A_i} \eta_i]. \end{aligned}$$

It follows that  $\hat{\mathbb{E}}[\sum_{i=1}^N I_{A_i} \hat{\mathbb{E}}_t[\eta_i]] \leq \hat{\mathbb{E}}[\sum_{i=1}^N I_{A_i} \eta_i]$ . Similarly we can prove  $\hat{\mathbb{E}}[\sum_{i=1}^N I_{A_i} \eta_i] \leq \hat{\mathbb{E}}[\sum_{i=1}^N I_{A_i} \hat{\mathbb{E}}_t[\eta_i]]$ .  $\square$

Let  $\mathbb{L}_G^{p,t}(\Omega_T)$  be the completion of  $\mathbb{L}_G^{0,p,t}(\Omega_T)$  under the norm  $(\hat{\mathbb{E}}[|\cdot|^p])^{1/p}$ . Clearly, the conditional  $G$ -expectation can be extended continuously to  $\mathbb{L}_G^{p,t}(\Omega_T)$ .

Set

$$\mathbb{M}^{p,0}(0, T) := \{\eta_t = \sum_{i=0}^{N-1} \xi_{t_i} I_{[t_i, t_{i+1})}(t) : 0 = t_0 < \dots < t_N = T, \xi_{t_i} \in \mathbb{L}^p(\Omega_{t_i})\}.$$

For  $p \geq 1$ , we denote by  $\mathbb{M}^p(0, T)$ ,  $\mathbb{H}^p(0, T)$ ,  $\mathbb{S}^p(0, T)$  the completion of  $\mathbb{M}^{p,0}(0, T)$  under the norm  $\|\eta\|_{\mathbb{M}^p} := (\hat{\mathbb{E}}[\int_0^T |\eta_t|^p dt])^{1/p}$ ,  $\|\eta\|_{\mathbb{H}^p} := \{\hat{\mathbb{E}}[(\int_0^T |\eta_t|^2 dt)^{p/2}]\}^{1/p}$ ,  $\|\eta\|_{\mathbb{D}^p} := (\hat{\mathbb{E}}[\sup_{t \in [0, T]} |\eta_t|^p])^{1/p}$  respectively. Following Li and Peng [9], for each  $\eta \in \mathbb{H}^p(0, T)$  with  $p \geq 1$ , we can define Itô's integral  $\int_0^T \eta_s dB_s$ . Moreover, by Proposition 2.10 in [9] and classical Burkholder-Davis-Gundy Inequality, the following properties hold.

**Proposition 2.11** *For each  $\eta, \theta \in \mathbb{H}^\alpha(0, T)$  with  $\alpha \geq 1$  and  $p > 0$ ,  $\xi \in \mathbb{L}^\infty(\Omega_t)$ , we have*

$$\begin{aligned} \hat{\mathbb{E}}[\int_0^T \eta_s dB_s] &= 0, \\ \underline{c}^p c_p \hat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}] &\leq \hat{\mathbb{E}}[\sup_{t \in [0, T]} |\int_0^t \eta_s dB_s|^p] \leq \bar{\sigma}^p C_p \hat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}], \\ \int_t^T (\xi \eta_s + \theta_s) dB_s &= \xi \int_t^T \eta_s dB_s + \int_t^T \theta_s dB_s, \end{aligned}$$

where  $0 < c_p < C_p < \infty$  are constants.

**Definition 2.12** *A process  $\{M_t\}$  with values in  $L_G^1(\Omega_T)$  is called a  $G$ -martingale if  $\hat{\mathbb{E}}_s[M_t] = M_s$  for any  $s \leq t$ .*

For  $\xi \in L_{ip}(\Omega_T)$ , let  $\mathcal{E}[\xi] = \hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[\xi]]$ , where  $\hat{\mathbb{E}}$  is the  $G$ -expectation. For convenience, we call  $\mathcal{E}$   $G$ -evaluation.

For  $p \geq 1$  and  $\xi \in L_{ip}(\Omega_T)$ , define  $\|\xi\|_{p, \mathcal{E}} = \{\mathcal{E}[|\xi|^p]\}^{1/p}$  and denote by  $L_{\mathcal{E}}^p(\Omega_T)$  the completion of  $L_{ip}(\Omega_T)$  under the norm  $\|\cdot\|_{p, \mathcal{E}}$ .

Let  $S_G^0(0, T) = \{h(t, B_{t_1 \wedge t}, \dots, B_{t_n \wedge t}) : t_1, \dots, t_n \in [0, T], h \in C_{b, Lip}(\mathbb{R}^{n+1})\}$ . For  $p \geq 1$  and  $\eta \in S_G^0(0, T)$ , set  $\|\eta\|_{D_G^p} = \{\hat{\mathbb{E}}[\sup_{t \in [0, T]} |\eta_t|^p]\}^{\frac{1}{p}}$ . Denote by  $S_G^p(0, T)$  the completion of  $S_G^0(0, T)$  under the norm  $\|\cdot\|_{S_G^p}$ .

The following estimate will be frequently used in this paper.

**Theorem 2.13** ([27]) *For any  $\alpha \geq 1$  and  $\delta > 0$ , we have  $L_G^{\alpha+\delta}(\Omega_T) \subset L_{\mathcal{E}}^\alpha(\Omega_T)$ . More precisely, for any  $1 < \gamma < \beta := (\alpha + \delta)/\alpha$ ,  $\gamma \leq 2$  and for all  $\xi \in L_{ip}(\Omega_T)$ , we have*

$$\hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[|\xi|^\alpha]] \leq C\{(\hat{\mathbb{E}}[|\xi|^{\alpha+\delta}])^{\alpha/(\alpha+\delta)} + (\hat{\mathbb{E}}[|\xi|^{\alpha+\delta}])^{1/\gamma}\}, \quad (2.1)$$

where  $C = \frac{\gamma}{\gamma-1}(1 + 14 \sum_{i=1}^\infty i^{-\beta/\gamma})$ .

**Remark 2.14** *By  $\frac{\alpha}{\alpha+\delta} < \frac{1}{\gamma} < 1$ , we have*

$$\hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[|\xi|^\alpha]] \leq 2C\{(\hat{\mathbb{E}}[|\xi|^{\alpha+\delta}])^{\alpha/(\alpha+\delta)} + \hat{\mathbb{E}}[|\xi|^{\alpha+\delta}]\}.$$

Set  $C_1 = 2 \inf\{\frac{\gamma}{\gamma-1}(1 + 14 \sum_{i=1}^{\infty} i^{-\beta/\gamma}) : 1 < \gamma < \beta, \gamma \leq 2\}$ , then

$$\hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[|\xi|^\alpha]] \leq C_1 \{(\hat{\mathbb{E}}[|\xi|^{\alpha+\delta}])^{\alpha/(\alpha+\delta)} + \hat{\mathbb{E}}[|\xi|^{\alpha+\delta}]\}, \quad (2.2)$$

where  $C_1$  is a constant only depending on  $\alpha$  and  $\delta$ .

For readers' convenience, we list the main notations of this paper as follows:

- The scalar product and norm of the Euclid space  $\mathbb{R}^n$  are respectively denoted by  $\langle \cdot, \cdot \rangle$  and  $|\cdot|$ ;
- $L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \dots, B_{t_n}) : n \geq 1, t_1, \dots, t_n \in [0, T], \varphi \in C_{b, Lip}(\mathbb{R}^{d \times n})\}$ ;
- $\|\xi\|_{p, G} = (\hat{\mathbb{E}}[|\xi|^p])^{1/p}$ ,  $\|\xi\|_{p, \mathcal{E}} = (\hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[|\xi|^p]])^{1/p}$ ;
- $L_G^p(\Omega_T) :=$  the completion of  $L_{ip}(\Omega_T)$  under  $\|\cdot\|_{p, G}$ ;
- $L_{\mathcal{E}}^p(\Omega_T) :=$  the completion of  $L_{ip}(\Omega_T)$  under  $\|\cdot\|_{p, \mathcal{E}}$ ;
- $M_G^0(0, T) := \{\eta_t = \sum_{j=0}^{N-1} \xi_j I_{[t_j, t_{j+1})}(t) : 0 = t_0 < \dots < t_N = T, \xi_i \in L_{ip}(\Omega_{t_i})\}$ ;
- $\|\eta\|_{M_G^p} = \{\hat{\mathbb{E}}[\int_0^T |\eta_s|^p ds]\}^{1/p}$ ,  $\|\eta\|_{H_G^p} = \{\hat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}]\}^{1/p}$ ;
- $M_G^p(0, T) :=$  the completion of  $M_G^0(0, T)$  under  $\|\cdot\|_{M_G^p}$ ;
- $H_G^p(0, T) :=$  the completion of  $M_G^0(0, T)$  under  $\|\cdot\|_{H_G^p}$  for  $p \geq 1$ ;
- $\mathbb{L}^p(\Omega_T) := \{X \in \mathcal{B}(\Omega_T) : \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty\}$  for  $p \geq 1$ ;
- $\mathbb{M}^{p,0}(0, T) := \{\eta_t = \sum_{i=0}^{N-1} \xi_{t_i} I_{[t_i, t_{i+1})}(t) : 0 = t_0 < \dots < t_N = T, \xi_{t_i} \in \mathbb{L}^p(\Omega_{t_i})\}$ ;
- $\|\eta\|_{\mathbb{M}^p} = (\hat{\mathbb{E}}[\int_0^T |\eta_t|^p dt])^{1/p}$ ,  $\|\eta\|_{\mathbb{H}^p} = \{\hat{\mathbb{E}}[(\int_0^T |\eta_t|^2 dt)^{p/2}]\}^{1/p}$ ;
- $\|\eta\|_{\mathbb{S}^p} = \{\hat{\mathbb{E}}[\sup_{t \in [0, T]} |\eta_t|^p]\}^{\frac{1}{p}}$ ;
- $\mathbb{M}^p(0, T) :=$  the completion of  $\mathbb{M}^{p,0}(0, T)$  under  $\|\cdot\|_{\mathbb{M}^p}$ ;
- $\mathbb{H}^p(0, T) :=$  the completion of  $\mathbb{M}^{p,0}(0, T)$  under  $\|\cdot\|_{\mathbb{H}^p}$ ;
- $\mathbb{S}^p(0, T) :=$  the completion of  $\mathbb{M}^{p,0}(0, T)$  under  $\|\cdot\|_{\mathbb{S}^p}$ ;
- $S_G^0(0, T) = \{h(t, B_{t_1 \wedge t}, \dots, B_{t_n \wedge t}) : t_1, \dots, t_n \in [0, T], h \in C_{b, Lip}(\mathbb{R}^{n+1})\}$ ;
- $\|\eta\|_{S_G^p} = \{\hat{\mathbb{E}}[\sup_{t \in [0, T]} |\eta_t|^p]\}^{\frac{1}{p}}$ ;
- $S_G^p(0, T) :=$  the completion of  $S_G^0(0, T)$  under  $\|\cdot\|_{S_G^p}$ ;
- $\mathfrak{S}_G^\alpha(0, T) :=$  the collection of processes  $(Y, Z, K)$  such that  $Y \in S_G^\alpha(0, T)$ ,  $Z \in H_G^\alpha(0, T)$ ,  $K$  is a decreasing  $G$ -martingale with  $K_0 = 0$  and  $K_T \in L_G^\alpha(\Omega_T)$ .

### 3 A priori estimates

For simplicity, we consider the  $G$ -expectation space  $(\Omega_T, L_G^1(\Omega_T), \hat{\mathbb{E}})$  with  $\Omega_T = C_0([0, T], \mathbb{R})$  and  $\bar{\sigma}^2 = \hat{\mathbb{E}}[B_1^2] \geq -\hat{\mathbb{E}}[-B_1^2] = \underline{\sigma}^2 > 0$ . But our results and methods still hold for the case  $d > 1$ .

We consider the following type of  $G$ -BSDEs for simplicity, and similar estimates hold for equation (1.2).

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s - (K_T - K_t), \quad (3.1)$$

where

$$f(t, \omega, y, z) : [0, T] \times \Omega_T \times \mathbb{R}^2 \rightarrow \mathbb{R}$$

satisfies the following properties: There exists some  $\beta > 1$  such that

(H1) for any  $y, z$ ,  $f(\cdot, \cdot, y, z) \in M_G^\beta(0, T)$ ;

(H2)  $|f(t, \omega, y, z) - f(t, \omega, y', z')| \leq L(|y - y'| + |z - z'|)$  for some  $L > 0$ .

For simplicity, we denote by  $\mathfrak{S}_G^\alpha(0, T)$  the collection of processes  $(Y, Z, K)$  such that  $Y \in S_G^\alpha(0, T)$ ,  $Z \in H_G^\alpha(0, T)$ ,  $K$  is a decreasing  $G$ -martingale with  $K_0 = 0$  and  $K_T \in L_G^\alpha(\Omega_T)$ .

**Definition 3.1** Let  $\xi \in L_G^\beta(\Omega_T)$  with  $\beta > 1$  and  $f$  satisfy (H1) and (H2). A triplet of processes  $(Y, Z, K)$  is called a solution of equation (3.1) if for some  $1 < \alpha \leq \beta$  the following properties hold:

(a)  $(Y, Z, K) \in \mathfrak{S}_G^\alpha(0, T)$ ;

(b)  $Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s - (K_T - K_t)$ .

In order to get a priori estimates for the solution of equation (3.1), we need the following lemmas.

**Lemma 3.2** Let  $X \in S_G^\alpha(0, T)$  for some  $\alpha > 1$ . Set

$$X_t^n = \sum_{i=0}^{n-1} X_{t_i^n} I_{[t_i^n, t_{i+1}^n)}(t),$$

where  $t_i^n = \frac{iT}{n}$ ,  $i = 0, \dots, n$ . Then

$$\hat{\mathbb{E}}\left[\sup_{t \in [0, T]} |X_t^n - X_t|^\alpha\right] \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.2)$$

**Proof.** For each given  $n$ ,  $m \geq 1$ , it is easy to check that

$$\sup_{i \leq n-1} \sup_{t_k^m \in [t_i^n, t_{i+1}^n]} |B_{t_k^m} - B_{t_i^n}|^\alpha$$

is a convex function, then by Proposition 11 in Peng [17], we get

$$\hat{\mathbb{E}}\left[\sup_{i \leq n-1} \sup_{t_k^n \in [t_i^n, t_{i+1}^n]} |B_{t_k^n}^m - B_{t_i^n}^m|^\alpha\right] = E_{P_{\bar{\sigma}}}\left[\sup_{i \leq n-1} \sup_{t_k^n \in [t_i^n, t_{i+1}^n]} |B_{t_k^n}^m - B_{t_i^n}^m|^\alpha\right],$$

where  $P_{\bar{\sigma}}$  is a Wiener measure on  $\Omega_T$  such that  $E_{P_{\bar{\sigma}}}[B_1^2] = \bar{\sigma}^2$ . Noting that

$$\sup_{i \leq n-1} \sup_{t_k^n \in [t_i^n, t_{i+1}^n]} |B_{t_k^n}^m - B_{t_i^n}^m|^\alpha \uparrow \sup_{i \leq n-1} \sup_{t \in [t_i^n, t_{i+1}^n]} |B_t - B_{t_i^n}|^\alpha \text{ as } m \uparrow \infty,$$

we have

$$\hat{\mathbb{E}}\left[\sup_{i \leq n-1} \sup_{t \in [t_i^n, t_{i+1}^n]} |B_t - B_{t_i^n}|^\alpha\right] = E_{P_{\bar{\sigma}}}\left[\sup_{i \leq n-1} \sup_{t \in [t_i^n, t_{i+1}^n]} |B_t - B_{t_i^n}|^\alpha\right] \rightarrow 0.$$

From this we can get  $\hat{\mathbb{E}}[\sup_{t \in [0, T]} |\eta_t - \eta_t^n|^\alpha] \rightarrow 0$  for each  $\eta \in S_G^0(0, T)$ . By the definition of  $S_G^\alpha(0, T)$ , we can choose a sequence  $(\eta^m)_{m=1}^\infty \subset S_G^0(0, T)$  such that  $\hat{\mathbb{E}}[\sup_{t \in [0, T]} |X_t - \eta_t^m|^\alpha] \rightarrow 0$  as  $m \rightarrow \infty$ . Note that

$$\sup_{t \in [0, T]} |X_t - X_t^n| \leq 2 \sup_{t \in [0, T]} |X_t - \eta_t^m| + \sup_{t \in [0, T]} |\eta_t^m - (\eta^m)_t^n|,$$

then we obtain (3.2) by letting  $n \rightarrow \infty$  first and then  $m \rightarrow \infty$ .  $\square$

**Lemma 3.3** *Let  $X_t, X_t^n$  be as in Lemma 3.2 and  $\alpha^* = \frac{\alpha}{\alpha-1}$ . Assume that  $K$  is a decreasing  $G$ -martingale with  $K_0 = 0$  and  $K_T \in L_G^{\alpha^*}(\Omega_T)$ . Then we have*

$$\hat{\mathbb{E}}\left[\sup_{t \in [0, T]} \left| \int_0^t X_s^n dK_s - \int_0^t X_s dK_s \right|\right] \rightarrow 0 \text{ as } n \rightarrow \infty.$$

**Proof.**

$$\begin{aligned} & \sup_{t \in [0, T]} \left| \int_0^t X_s^n dK_s - \int_0^t X_s dK_s \right| \\ & \leq - \int_0^T |X_s^n - X_s| dK_s \\ & \leq \sup_{s \in [0, T]} |X_s^n - X_s| (-K_T). \end{aligned}$$

So we have

$$\hat{\mathbb{E}}\left[\sup_{t \in [0, T]} \left| \int_0^t X_s^n dK_s - \int_0^t X_s dK_s \right|\right] \leq \left\| \sup_{s \in [0, T]} |X_s^n - X_s| \right\|_{L_G^\alpha} \|K_T\|_{L^{\alpha^*}} \rightarrow 0$$

as  $n \rightarrow \infty$ .  $\square$

**Lemma 3.4** *Let  $X \in S_G^\alpha(0, T)$  for some  $\alpha > 1$  and  $\alpha^* = \frac{\alpha}{\alpha-1}$ . Assume that  $K^j, j = 1, 2$ , are two decreasing  $G$ -martingales with  $K_0^j = 0$  and  $K_T^j \in L_G^{\alpha^*}(\Omega_T)$ . Then the process defined by*

$$\int_0^t X_s^+ dK_s^1 + \int_0^t X_s^- dK_s^2$$

*is also a decreasing  $G$ -martingale.*

**Proof.** Let  $X^n$  be as in Lemma 3.2. By Lemma 3.3, it suffices to prove that the process

$$\int_0^t (X_s^n)^+ dK_s^1 + \int_0^t (X_s^n)^- dK_s^2$$

is a  $G$ -martingale. By properties of conditional  $G$ -expectation, we have, for any  $t \in [t_i^n, t_{i+1}^n]$ ,

$$\begin{aligned} & \hat{\mathbb{E}}_t[X_{t_i^n}^+(K_{t_{i+1}^n}^1 - K_{t_i^n}^1) + X_{t_i^n}^-(K_{t_{i+1}^n}^2 - K_{t_i^n}^2)] \\ &= X_{t_i^n}^+ \hat{\mathbb{E}}_t[K_{t_{i+1}^n}^1 - K_{t_i^n}^1] + X_{t_i^n}^- \hat{\mathbb{E}}_t[K_{t_{i+1}^n}^2 - K_{t_i^n}^2] \\ &= X_{t_i^n}^+(K_t^1 - K_{t_i^n}^1) + X_{t_i^n}^-(K_t^2 - K_{t_i^n}^2). \end{aligned}$$

From this we obtain that  $\int_0^t (X_s^n)^+ dK_s^1 + \int_0^t (X_s^n)^- dK_s^2$  is a  $G$ -martingale.  $\square$

Now we give a priori estimates for the solution of equation (3.1). For this purpose, a weaker version of condition (H2) is enough.

**(H2')**  $|f(t, \omega, y, z) - f(t, \omega, y', z')| \leq L^w(|y - y'| + |z - z'| + \varepsilon)$  for some  $L^w, \varepsilon > 0$ .

In the following three propositions,  $C_\alpha$  will always designate a constant depending on  $\alpha, T, L^w, \underline{\sigma}$ , which may vary from line to line.

**Proposition 3.5** *Let  $f$  satisfy (H1) and (H2'). Assume*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s - (K_T - K_t),$$

where  $Y \in \mathbb{S}^\alpha(0, T)$ ,  $Z \in \mathbb{H}^\alpha(0, T)$ ,  $K$  is a decreasing process with  $K_0 = 0$  and  $K_T \in \mathbb{L}^\alpha(\Omega_T)$  for some  $\alpha > 1$ . Then there exists a constant  $C_\alpha := C(\alpha, T, \underline{\sigma}, L^w) > 0$  such that

$$\hat{\mathbb{E}}[(\int_0^T |Z_s|^2 ds)^{\frac{\alpha}{2}}] \leq C_\alpha \{ \hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t|^\alpha] + (\hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t|^\alpha])^{\frac{1}{2}} (\hat{\mathbb{E}}[(\int_0^T f_s^0 ds)^\alpha])^{\frac{1}{2}} \}, \quad (3.3)$$

$$\hat{\mathbb{E}}[|K_T|^\alpha] \leq C_\alpha \{ \hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t|^\alpha] + \hat{\mathbb{E}}[(\int_0^T f_s^0 ds)^\alpha] \}, \quad (3.4)$$

where  $f_s^0 = |f(s, 0, 0)| + L^w \varepsilon$ .

**Proof.** Applying Itô's formula to  $|Y_t|^2$ , we have

$$|Y_0|^2 + \int_0^T |Z_s|^2 d\langle B \rangle_s = |\xi|^2 + \int_0^T 2Y_s f(s) ds - \int_0^T 2Y_s Z_s dB_s - \int_0^T 2Y_s dK_s,$$

where  $f(s) = f(s, Y_s, Z_s)$ . Then

$$(\int_0^T |Z_s|^2 d\langle B \rangle_s)^{\frac{\alpha}{2}} \leq C_\alpha \{ |\xi|^\alpha + |\int_0^T Y_s f(s) ds|^{\frac{\alpha}{2}} + |\int_0^T Y_s Z_s dB_s|^{\frac{\alpha}{2}} + |\int_0^T Y_s dK_s|^{\frac{\alpha}{2}} \}.$$

By Proposition 2.11 and simple calculation, we can obtain

$$\hat{\mathbb{E}}[(\int_0^T |Z_s|^2 ds)^{\frac{\alpha}{2}}] \leq C_\alpha \{ \|Y\|_{\mathbb{S}^\alpha}^\alpha + \|Y\|_{\mathbb{S}^\alpha}^{\frac{\alpha}{2}} [(\hat{\mathbb{E}}[|K_T|^\alpha])^{\frac{1}{2}} + (\hat{\mathbb{E}}[(\int_0^T f_s^0 ds)^\alpha])^{\frac{1}{2}}] \}. \quad (3.5)$$

On the other hand,

$$K_T = \xi - Y_0 + \int_0^T f(s) ds - \int_0^T Z_s dB_s.$$

By simple calculation, we get

$$\hat{\mathbb{E}}[|K_T|^\alpha] \leq C_\alpha \{ \|Y\|_{\mathbb{S}^\alpha}^\alpha + \hat{\mathbb{E}}[(\int_0^T |Z_s|^2 ds)^{\alpha/2}] + \hat{\mathbb{E}}[(\int_0^T f_s^0 ds)^\alpha] \}. \quad (3.6)$$

By (3.5) and (3.6), it is easy to get (3.3) and (3.4).  $\square$

**Remark 3.6** *In this proposition, we do not assume that  $(Y, Z, K)$  belong to  $\mathfrak{S}_G^\alpha(0, T)$ .*

**Proposition 3.7** *Let  $\xi \in L_G^\beta(\Omega_T)$  with  $\beta > 1$  and  $f$  satisfy (H1) and (H2'). Assume that  $(Y, Z, K) \in \mathfrak{S}_G^\alpha(0, T)$  for some  $1 < \alpha < \beta$  is a solution of equation (3.1). Then*

(i) *There exists a constant  $C_\alpha := C(\alpha, T, \underline{\sigma}, L^w) > 0$  such that*

$$|Y_t|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t[|\xi|^\alpha + \int_t^T |f_s^0|^\alpha ds], \quad (3.7)$$

$$\hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t|^\alpha] \leq C_\alpha \hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[|\xi|^\alpha + \int_0^T |f_s^0|^\alpha ds]], \quad (3.8)$$

where  $f_s^0 = |f(s, 0, 0)| + L^w \varepsilon$ .

(ii) *For any given  $\alpha'$  with  $\alpha < \alpha' < \beta$ , there exists a constant  $C_{\alpha, \alpha'}$  depending on  $\alpha, \alpha', T, \underline{\sigma}, L^w$  such that*

$$\begin{aligned} \hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t|^\alpha] &\leq C_{\alpha, \alpha'} \{ \hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[|\xi|^\alpha]] \\ &\quad + (\hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[(\int_0^T f_s^0 ds)^{\alpha'}])^{\frac{\alpha}{\alpha'}} + \hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[(\int_0^T f_s^0 ds)^{\alpha'}]] \}. \end{aligned} \quad (3.9)$$

**Proof.** For any  $\gamma, \epsilon > 0$ , set  $\tilde{Y}_t = |Y_t|^2 + \epsilon_\alpha$ , where  $\epsilon_\alpha = \epsilon(1 - \alpha/2)^+$ , applying Itô's formula to  $\tilde{Y}_t^{\alpha/2} e^{\gamma t}$ , we have

$$\begin{aligned}
& \tilde{Y}_t^{\alpha/2} e^{\gamma t} + \gamma \int_t^T e^{\gamma s} \tilde{Y}_s^{\alpha/2} ds + \frac{\alpha}{2} \int_t^T e^{\gamma s} \tilde{Y}_s^{\alpha/2-1} Z_s^2 d\langle B \rangle_s \\
&= (|\xi|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma T} + \alpha(1 - \frac{\alpha}{2}) \int_t^T e^{\gamma s} \tilde{Y}_s^{\alpha/2-2} Y_s^2 Z_s^2 d\langle B \rangle_s \\
&+ \int_t^T \alpha e^{\gamma s} \tilde{Y}_s^{\alpha/2-1} Y_s f(s) ds - \int_t^T \alpha e^{\gamma s} \tilde{Y}_s^{\alpha/2-1} (Y_s Z_s dB_s + Y_s dK_s) \\
&\leq (|\xi|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma T} + \alpha(1 - \frac{\alpha}{2}) \int_t^T e^{\gamma s} \tilde{Y}_s^{\alpha/2-1} Z_s^2 d\langle B \rangle_s \\
&+ \int_t^T \alpha e^{\gamma s} \tilde{Y}_s^{\alpha/2-1/2} |f(s)| ds - (M_T - M_t), \tag{3.10}
\end{aligned}$$

where  $f(s) = f(s, Y_s, Z_s)$  and

$$M_t = \int_0^t \alpha e^{\gamma s} \tilde{Y}_s^{\alpha/2-1} Y_s Z_s dB_s + \int_0^t \alpha e^{\gamma s} \tilde{Y}_s^{\alpha/2-1} Y_s^+ dK_s.$$

From the assumption of  $f$ , we have

$$\begin{aligned}
& \int_t^T \alpha e^{\gamma s} \tilde{Y}_s^{\alpha/2-1/2} |f(s)| ds \\
&\leq \int_t^T \alpha e^{\gamma s} \tilde{Y}_s^{\alpha/2-1/2} (f_s^0 + L^w |Y_s| + L^w |Z_s|) ds \\
&\leq (\alpha L^w + \frac{\alpha(L^w)^2}{\underline{\alpha}^2(\alpha-1)}) \int_t^T e^{\gamma s} \tilde{Y}_s^{\alpha/2} ds + \frac{\alpha(\alpha-1)}{4} \int_t^T e^{\gamma s} \tilde{Y}_s^{\alpha/2-1} Z_s^2 d\langle B \rangle_s \\
&+ \int_t^T \alpha e^{\gamma s} \tilde{Y}_s^{\alpha/2-1/2} |f_s^0| ds. \tag{3.11}
\end{aligned}$$

(i) By Young's inequality, we have

$$\int_t^T \alpha e^{\gamma s} \tilde{Y}_s^{\alpha/2-1/2} |f_s^0| ds \leq (\alpha-1) \int_t^T e^{\gamma s} \tilde{Y}_s^{\alpha/2} ds + \int_t^T e^{\gamma s} |f_s^0|^\alpha ds. \tag{3.12}$$

By (3.10), (3.11) and (3.12), we have

$$\begin{aligned}
& \tilde{Y}_t^{\alpha/2} e^{\gamma t} + (\gamma - \tilde{\alpha}) \int_t^T e^{\gamma s} \tilde{Y}_s^{\alpha/2} ds + \frac{\alpha(\alpha-1)}{4} \int_t^T e^{\gamma s} \tilde{Y}_s^{\alpha/2-1} Z_s^2 d\langle B \rangle_s \\
&\leq (|\xi|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma T} + \int_t^T e^{\gamma s} |f_s^0|^\alpha ds - (M_T - M_t),
\end{aligned}$$

where  $\tilde{\alpha} = \alpha L^w + \alpha + \frac{\alpha(L^w)^2}{\underline{\alpha}^2(\alpha-1)} - 1$ . Setting  $\gamma = \tilde{\alpha} + 1$ , we have

$$\begin{aligned}
& \tilde{Y}_t^{\alpha/2} e^{\gamma t} + M_T - M_t \\
&\leq (|\xi|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma T} + \int_t^T e^{\gamma s} |f_s^0|^\alpha ds.
\end{aligned}$$



By Lemma 3.4,  $M_t$  is a  $G$ -martingale, so we have

$$\tilde{Y}_t^{\alpha/2} e^{\gamma t} \leq \hat{\mathbb{E}}_t[(|\xi|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma T} + \int_t^T e^{\gamma s} |f_s^0|^\alpha ds].$$

By letting  $\epsilon \downarrow 0$ , there exists a constant  $C_\alpha := C_\alpha(T, L^w, \underline{g})$  such that

$$|Y_t|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t[|\xi|^\alpha + \int_t^T |f_s^0|^\alpha ds].$$

It follows that

$$\hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t|^\alpha] \leq C_\alpha \hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[|\xi|^\alpha + \int_0^T |f_s^0|^\alpha ds]].$$

(ii) By (3.10) and (3.11) and setting  $\gamma = \alpha L^w + \frac{\alpha(L^w)^2}{\underline{g}^2(\alpha-1)} + 1$ , then we get

$$\tilde{Y}_t^{\alpha/2} e^{\gamma t} \leq \hat{\mathbb{E}}_t[(|\xi|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma T} + \int_t^T \alpha e^{\gamma s} \tilde{Y}_s^{\alpha/2-1/2} f_s^0 ds].$$

By letting  $\epsilon \downarrow 0$ , we get

$$|Y_t|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t[|\xi|^\alpha + \int_t^T |Y_s|^{\alpha-1} f_s^0 ds]. \quad (3.13)$$

From this we get

$$\begin{aligned} |Y_t|^\alpha &\leq C_\alpha \{ \hat{\mathbb{E}}_t[|\xi|^\alpha] + \hat{\mathbb{E}}_t[ \sup_{s \in [0, T]} |Y_s|^{\alpha-1} \int_0^T f_s^0 ds] \} \\ &\leq C_\alpha \{ \hat{\mathbb{E}}_t[|\xi|^\alpha] + (\hat{\mathbb{E}}_t[ \sup_{s \in [0, T]} |Y_s|^{(\alpha-1)\alpha'^*}])^{\frac{1}{\alpha'^*}} (\hat{\mathbb{E}}_t[(\int_0^T f_s^0 ds)^{\alpha'}])^{\frac{1}{\alpha'}}, \end{aligned} \quad (3.14)$$

where  $\alpha'^* = \frac{\alpha'}{\alpha'-1}$ . Thus we obtain

$$\hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t|^\alpha] \leq C_\alpha \{ \|\xi\|_{\alpha, \mathcal{E}}^\alpha + \|\sup_{s \in [0, T]} |Y_s|^{\alpha-1} \int_0^T f_s^0 ds\|_{\alpha', \mathcal{E}} \}.$$

It is easy to check that  $(\alpha-1)\alpha'^* < \alpha$ , then by (2.2) there exists a constant  $C$  only depending on  $\alpha$  and  $\alpha'$  such that

$$\|\sup_{s \in [0, T]} |Y_s|^{\alpha-1} \int_0^T f_s^0 ds\|_{\alpha', \mathcal{E}} \leq C \{ (\hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t|^\alpha])^{\frac{\alpha-1}{\alpha}} + (\hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t|^\alpha])^{\frac{1}{\alpha'^*}} \}.$$

By Young's inequality, we have

$$CC_\alpha (\hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t|^\alpha])^{\frac{\alpha-1}{\alpha}} \|\int_0^T f_s^0 ds\|_{\alpha', \mathcal{E}} \leq \frac{1}{4} \hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t|^\alpha] + C_1 C_\alpha \|\int_0^T f_s^0 ds\|_{\alpha_1, \mathcal{E}}^\alpha$$

and

$$CC_\alpha (\hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t|^\alpha])^{\frac{1}{\alpha'^*}} \|\int_0^T f_s^0 ds\|_{\alpha_1, \mathcal{E}} \leq \frac{1}{4} \hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t|^\alpha] + C_1 C_\alpha \|\int_0^T f_s^0 ds\|_{\alpha_1, \mathcal{E}}^{\alpha'},$$

where  $C_1$  is a constant only depending on  $\alpha$  and  $\alpha'$ . Thus we obtain (3.9).  $\square$

**Proposition 3.8** Let  $f_i$ ,  $i = 1, 2$ , satisfy (H1) and (H2'). Assume

$$Y_t^i = \xi^i + \int_t^T f_i(s, Y_s^i, Z_s^i) ds - \int_t^T Z_s^i dB_s - (K_T^i - K_t^i),$$

where  $Y^i \in \mathbb{S}^\alpha(0, T)$ ,  $Z^i \in \mathbb{H}^\alpha(0, T)$ ,  $K^i$  is a decreasing process with  $K_0^i = 0$  and  $K_T^i \in \mathbb{L}^\alpha(\Omega_T)$  for some  $\alpha > 1$ . Set  $\hat{Y}_t = Y_t^1 - Y_t^2$ ,  $\hat{Z}_t = Z_t^1 - Z_t^2$  and  $\hat{K}_t = K_t^1 - K_t^2$ . Then there exists a constant  $C_\alpha := C(\alpha, T, \underline{\sigma}, L^w) > 0$  such that

$$\hat{\mathbb{E}}[(\int_0^T |\hat{Z}_s|^2 ds)^{\frac{\alpha}{2}}] \leq C_\alpha \{ \|\hat{Y}\|_{\mathbb{S}^\alpha}^\alpha + \|\hat{Y}\|_{\mathbb{S}^\alpha}^{\frac{\alpha}{2}} \sum_{i=1}^2 [\|Y^i\|_{\mathbb{S}^\alpha}^{\frac{\alpha}{2}} + \|\int_0^T f_s^{i,0} ds\|_{\alpha,G}^{\frac{\alpha}{2}}] \}, \quad (3.15)$$

where  $f_s^{i,0} = |f_i(s, 0, 0)| + L^w \varepsilon$ ,  $i = 1, 2$ .

**Proof.** Applying Itô's formula to  $|\hat{Y}_t|^2$ , by similar analysis as that in Proposition 3.5, we have

$$\|\hat{Z}\|_{\mathbb{H}^\alpha}^\alpha \leq C_\alpha \{ \|\hat{Y}\|_{\mathbb{S}^\alpha}^\alpha + \|\hat{Y}\|_{\mathbb{D}^\alpha}^{\frac{\alpha}{2}} [\|K_T^1\|_{\alpha,G}^{\frac{\alpha}{2}} + \|K_T^2\|_{\alpha,G}^{\frac{\alpha}{2}} + \|\int_0^T \hat{f}_s ds\|_{\alpha,G}^{\frac{\alpha}{2}}] \},$$

where  $\hat{f}_s = |f_1(s, Y_s^2, Z_s^2) - f_2(s, Y_s^2, Z_s^2)| + L^w \varepsilon$ . By Proposition 3.5, we obtain

$$\begin{aligned} & \|K_T^1\|_{\alpha,G}^{\frac{\alpha}{2}} + \|K_T^2\|_{\alpha,G}^{\frac{\alpha}{2}} + \|\int_0^T \hat{f}_s ds\|_{\alpha,G}^{\frac{\alpha}{2}} \\ & \leq C_\alpha \{ \|Y^1\|_{\mathbb{S}^\alpha}^{\frac{\alpha}{2}} + \|Y^2\|_{\mathbb{S}^\alpha}^{\frac{\alpha}{2}} + \|\int_0^T f_s^{1,0} ds\|_{\alpha,G}^{\frac{\alpha}{2}} + \|\int_0^T f_s^{2,0} ds\|_{\alpha,G}^{\frac{\alpha}{2}} \}. \end{aligned}$$

Thus we get (3.15).  $\square$

**Proposition 3.9** Let  $\xi^i \in L_G^\beta(\Omega_T)$  with  $\beta > 1$ ,  $i = 1, 2$ , and  $f_i$  satisfy (H1) and (H2'). Assume that  $(Y^i, Z^i, K^i) \in \mathfrak{S}_G^\alpha(0, T)$  for some  $1 < \alpha < \beta$  are the solutions of equation (3.1) corresponding to  $\xi^i$  and  $f_i$ . Set  $\hat{Y}_t = Y_t^1 - Y_t^2$ ,  $\hat{Z}_t = Z_t^1 - Z_t^2$  and  $\hat{K}_t = K_t^1 - K_t^2$ . Then

(i) There exists a constant  $C_\alpha := C(\alpha, T, \underline{\sigma}, L_1^w) > 0$  such that

$$|\hat{Y}_t|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t[|\hat{\xi}|^\alpha + \int_t^T |\hat{f}_s|^\alpha ds], \quad (3.16)$$

where  $\hat{f}_s = |f_1(s, Y_s^2, Z_s^2) - f_2(s, Y_s^2, Z_s^2)| + L_1^w \varepsilon$ .

(ii) For any given  $\alpha'$  with  $\alpha < \alpha' < \beta$ , there exists a constant  $C_{\alpha,\alpha'}$  depending on  $\alpha, \alpha', T, \underline{\sigma}, L^w$  such that

$$\begin{aligned} \hat{\mathbb{E}}[\sup_{t \in [0, T]} |\hat{Y}_t|^\alpha] & \leq C_{\alpha,\alpha'} \{ \hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[|\hat{\xi}|^\alpha]] \\ & + (\hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[(\int_0^T \hat{f}_s ds)^{\alpha'}]]^{\frac{\alpha}{\alpha'}} + \hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[(\int_0^T \hat{f}_s ds)^{\alpha'}]] \}. \end{aligned} \quad (3.17)$$

**Proof.** For any  $\gamma, \epsilon > 0$ , applying Itô's formula to  $(|\hat{Y}_t|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma t}$ , where  $\epsilon_\alpha = \epsilon(1 - \alpha/2)^+$ , by similar analysis as in Proposition 3.7, we have by setting  $\gamma = \alpha L^w + \alpha + \frac{\alpha(L^w)^2}{\underline{\sigma}^2(\alpha-1)}$

$$\begin{aligned} & (|\hat{Y}_t|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma t} + \int_t^T \alpha e^{\gamma s} (|\hat{Y}_s|^2 + \epsilon_\alpha)^{\alpha/2-1} \hat{Y}_s \hat{Z}_s dB_s + J_T - J_t \\ & \leq (|\hat{\xi}|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma T} + \int_t^T e^{\gamma s} |\hat{f}_s|^\alpha ds \end{aligned}$$

and

$$\begin{aligned} & (|\hat{Y}_t|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma t} + \int_t^T \alpha e^{\gamma s} (|\hat{Y}_s|^2 + \epsilon_\alpha)^{\alpha/2-1} \hat{Y}_s \hat{Z}_s dB_s + J_T - J_t \\ & \leq (|\hat{\xi}|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma T} + \int_t^T \alpha e^{\gamma s} (|\hat{Y}_s|^2 + \epsilon_\alpha)^{\alpha/2-1/2} \hat{f}_s ds, \end{aligned}$$

where

$$J_t = \int_0^t \alpha e^{\gamma s} (|\hat{Y}_s|^2 + \epsilon_\alpha)^{\alpha/2-1} (\hat{Y}_s^+ dK_s^1 + \hat{Y}_s^- dK_s^2).$$

By Lemma 3.4,  $J_t$  is a  $G$ -martingale. Taking conditional  $G$ -expectation and letting  $\epsilon \downarrow 0$ , we obtain a constant  $C_\alpha := C_\alpha(T, L_1^w, \underline{\sigma}) > 0$  such that

$$|\hat{Y}_t|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t[|\hat{\xi}|^\alpha + \int_t^T |\hat{f}_s|^\alpha ds]$$

and

$$|\hat{Y}_t|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t[|\hat{\xi}|^\alpha + \int_t^T |\hat{Y}_s|^{\alpha-1} \hat{f}_s ds].$$

By the same analysis as that in Proposition 3.7, we get (3.17).  $\square$

## 4 Existence and uniqueness of $G$ -BSDEs

In order to prove the existence of equation (3.1), we start with the simple case  $f(t, \omega, y, z) = h(y, z)$ ,  $\xi = \varphi(B_T)$ . Here  $h \in C_0^\infty(\mathbb{R}^2)$ ,  $\varphi \in C_{b, Lip}(\mathbb{R}^2)$ . For this case, we can obtain the solution of equation (3.1) from the following nonlinear partial differential equation:

$$\partial_t u + G(\partial_{xx}^2 u) + h(u, \partial_x u) = 0, u(T, x) = \varphi(x). \quad (4.1)$$

Then we approximate the solution of equation (3.1) with more complicated  $f$  by those of equations (3.1) with much simpler  $\{f_n\}$ . More precisely, assume that  $\|f_n - f\|_{M_G^\beta} \rightarrow 0$  and  $(Y^n, Z^n, K^n)$  is the solution of the following  $G$ -BSDE

$$Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, Z_s^n) ds - \int_t^T Z_s^n dB_s - (K_T^n - K_t^n).$$

We try to prove that  $(Y^n, Z^n, K^n)$  converges to  $(Y, Z, K)$  and  $(Y, Z, K)$  is the solution of the following  $G$ -BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s - (K_T - K_t).$$

One of the main results of this paper is

**Theorem 4.1** *Assume that  $\xi \in L_G^\beta(\Omega_T)$  for some  $\beta > 1$  and  $f$  satisfies (H1) and (H2). Then equation (3.1) has a unique solution  $(Y, Z, K)$ . Moreover, for any  $1 < \alpha < \beta$  we have  $Y \in S_G^\alpha(0, T)$ ,  $Z \in H_G^\alpha(0, T)$  and  $K_T \in L_G^\alpha(\Omega_T)$ .*

**Proof.** The uniqueness of the solution is a direct consequence of the a priori estimates in Proposition 3.8 and Proposition 3.9. By these estimates it also suffices to prove the existence for the case  $\xi \in L_{ip}(\Omega_T)$  and then pass to the limit for the general situation.

Step 1.  $f(t, \omega, y, z) = h(y, z)$  with  $h \in C_0^\infty(\mathbb{R}^2)$ .

Part 1. We first consider the case  $\xi = \varphi(B_T - B_{t_1})$  with  $\varphi \in C_{b, Lip}(\mathbb{R})$  and  $t_1 < T$ . Let  $u$  be the solution of equation (4.1) with terminal condition  $\varphi$ . By Theorem 6.4.3 in Krylov [8] (see also Theorem 4.4 in Appendix C in Peng [22]), there exists a constant  $\alpha \in (0, 1)$  such that for each  $\kappa > 0$ ,

$$\|u\|_{C^{1+\alpha/2, 2+\alpha}([0, T-\kappa] \times \mathbb{R})} < \infty.$$

Applying Itô's formula to  $u(t, B_t - B_{t_1})$  on  $[t_1, T - \kappa]$ , we get

$$\begin{aligned} u(t, B_t - B_{t_1}) = & u(T - \kappa, B_{T-\kappa} - B_{t_1}) + \int_t^{T-\kappa} h(u, \partial_x u)(s, B_s - B_{t_1}) ds \\ & - \int_t^{T-\kappa} \partial_x u(s, B_s - B_{t_1}) dB_s - (K_{T-\kappa} - K_t), \end{aligned} \quad (4.2)$$

where  $K_t = \frac{1}{2} \int_{t_1}^t \partial_{xx}^2 u(s, B_s - B_{t_1}) d\langle B \rangle_s - \int_{t_1}^t G(\partial_{xx}^2 u(s, B_s - B_{t_1})) ds$  is a non-increasing  $G$ -martingale. We now prove that there exists a constant  $L_1 > 0$  such that

$$|u(t, x) - u(s, y)| \leq L_1(\sqrt{|t - s|} + |x - y|), \quad t, s \in [0, T], x, y \in \mathbb{R}. \quad (4.3)$$

For each fixed  $x_0 \in \mathbb{R}$ , set  $\tilde{u}(t, x) = u(t, x + x_0)$ , it is easy to check that  $\tilde{u}$  is the solution of the following PDE:

$$\partial_t \tilde{u} + G(\partial_{xx}^2 \tilde{u}) + h(\tilde{u}, \partial_x \tilde{u}) = 0, \quad \tilde{u}(T, x) = \varphi(x + x_0). \quad (4.4)$$

Define  $\hat{u}(t, x) = u(t, x) + L_\varphi |x_0| \exp(L_h(T - t))$ , where  $L_\varphi$  and  $L_h$  are the Lipschitz constants of  $\varphi$  and  $h$  respectively, it is easy to verify that  $\hat{u}$  is a supersolution of PDE (4.4). Thus by comparison theorem (see Theorem 2.4 in Appendix C in Peng [22]) we get

$$u(t, x + x_0) \leq u(t, x) + L_\varphi |x_0| \exp(L_h(T - t)), \quad t \in [0, T], x \in \mathbb{R}.$$

Since  $x_0$  is arbitrary, we get  $|u(t, x) - u(t, y)| \leq \hat{L}|x - y|$ , where  $\hat{L} = L_\varphi \exp(L_h T)$ . From this we can get  $|\partial_x u(t, x)| \leq \hat{L}$  for each  $t \in [0, T]$ ,  $x \in \mathbb{R}$ . On the other hand, for each fixed  $\bar{t} < \hat{t} < T$  and  $x \in \mathbb{R}$ , applying Itô's formula to  $u(s, x + B_s - B_{\bar{t}})$  on  $[\bar{t}, \hat{t}]$ , we get

$$u(\bar{t}, x) = \hat{\mathbb{E}}[u(\hat{t}, x + B_{\hat{t}} - B_{\bar{t}}) + \int_{\bar{t}}^{\hat{t}} h(u, \partial_x u)(s, x + B_s - B_{\bar{t}}) ds].$$

From this we deduce that

$$|u(\bar{t}, x) - u(\hat{t}, x)| \leq \hat{\mathbb{E}}[\hat{L}|B_{\hat{t}} - B_{\bar{t}}| + \tilde{L}|\hat{t} - \bar{t}|] \leq (\hat{L}\bar{\sigma} + \tilde{L}\sqrt{T})\sqrt{|\hat{t} - \bar{t}|},$$

where  $\tilde{L} = \sup_{(x,y) \in \mathbb{R}^2} |h(x,y)|$ . Thus we get (4.3) by taking  $L_1 = \max\{\hat{L}, \hat{L}\bar{\sigma} + \tilde{L}\sqrt{T}\}$ . Letting  $\kappa \downarrow 0$  in equation (4.2), it is easy to verify that

$$\hat{\mathbb{E}}[|Y_{T-\kappa} - \xi|^2 + \int_{T-\kappa}^T |Z_t|^2 dt + (K_{T-\kappa} - K_T)^2] \rightarrow 0,$$

where  $Y_t = u(t, B_t - B_{t_1})$  and  $Z_t = \partial_x u(t, B_t - B_{t_1})$ . Thus  $(Y_t, Z_t, K_t)_{t \in [t_1, T]}$  is a solution of equation (3.1) with terminal value  $\xi = \varphi(B_T - B_{t_1})$ . Furthermore, it is easy to check that  $Y \in S_G^\alpha(t_1, T)$ ,  $Z \in H_G^\alpha(t_1, T)$  and  $K_T \in L_G^\alpha(\Omega_T)$  for any  $\alpha > 1$ .

Part 2. We now consider the case  $\xi = \psi(B_{t_1}, B_T - B_{t_1})$  with  $\psi \in C_{b,Lip}(\mathbb{R}^2)$ , and the more general case can be proved similarly. For each fixed  $x \in \mathbb{R}$ , let  $u(\cdot, x, \cdot)$  be the solution of equation (4.1) with terminal condition  $\psi(x, \cdot)$ . By Part 1, we have

$$\begin{aligned} u(t, x, B_t - B_{t_1}) &= u(T, x, B_T - B_{t_1}) + \int_t^T h(u, \partial_y u)(s, x, B_s - B_{t_1}) ds \\ &\quad - \int_t^T \partial_y u(s, x, B_s - B_{t_1}) dB_s - (K_T^x - K_t^x), \end{aligned} \quad (4.5)$$

where  $K_t^x = \frac{1}{2} \int_{t_1}^t \partial_{yy}^2 u(s, x, B_s - B_{t_1}) d\langle B \rangle_s - \int_{t_1}^t G(\partial_{yy}^2 u(s, x, B_s - B_{t_1})) ds$ . We replace  $x$  by  $B_{t_1}$  and get

$$Y_t = Y_T + \int_t^T h(Y_s, Z_s) ds - \int_t^T Z_s dB_s - (K_T - K_t),$$

where  $Y_t = u(t, B_{t_1}, B_t - B_{t_1})$ ,  $Z_t = \partial_y u(t, B_{t_1}, B_t - B_{t_1})$  and

$$K_t = \frac{1}{2} \int_{t_1}^t \partial_{yy}^2 u(s, B_{t_1}, B_s - B_{t_1}) d\langle B \rangle_s - \int_{t_1}^t G(\partial_{yy}^2 u(s, B_{t_1}, B_s - B_{t_1})) ds.$$

Now we are in a position to prove  $(Y, Z, K) \in \mathfrak{S}_G^\alpha(0, T)$ . We use the following argument, for each given  $n \in \mathbb{N}$ , by partition of unity theorem, there exist  $h_i^n \in C_0^\infty(\mathbb{R})$  with the diameter of support  $\lambda(\text{supp}(h_i^n)) < 1/n$ ,  $0 \leq h_i^n \leq 1$ ,  $I_{[-n,n]}(x) \leq \sum_{i=1}^{k_n} h_i^n \leq 1$ . Choose  $x_i^n$  such that  $h_i^n(x_i^n) > 0$ . Through equation (4.5), we have

$$Y_t^n = Y_T^n + \int_t^T \sum_{i=1}^n h(y_s^{n,i}, z_s^{n,i}) h_i^n(B_{t_1}) ds - \int_t^T Z_s^n dB_s - (K_T^n - K_t^n),$$

where  $y_t^{n,i} = u(t, x_i^n, B_t - B_{t_1})$ ,  $z_t^{n,i} = \partial_y u(t, x_i^n, B_t - B_{t_1})$ ,  $Y_t^n = \sum_{i=1}^n y_t^{n,i} h_i^n(B_{t_1})$ ,  $Z_t^n = \sum_{i=1}^n z_t^{n,i} h_i^n(B_{t_1})$  and  $K_t^n = \sum_{i=1}^n K_t^{x_i^n} h_i^n(B_{t_1})$ .

By the same analysis as that in Part 1, we can obtain a constant  $L_2 > 0$  such that for each  $t, s \in [0, T]$ ,  $x, x', y, y' \in \mathbb{R}$ ,

$$|u(t, x, y) - u(s, x', y')| \leq L_2(\sqrt{|t-s|} + |x-x'| + |y-y'|).$$

From this we get

$$\begin{aligned} |Y_t - Y_t^n| &\leq \sum_{i=1}^{k_n} h_i^n(B_{t_1}) |u(t, x_i^n, B_t - B_{t_1}) - u(t, B_{t_1}, B_t - B_{t_1})| + |Y_t| I_{[|B_{t_1}| > n]} \\ &\leq \frac{L_2}{n} + \frac{\|u\|_\infty}{n} |B_{t_1}|. \end{aligned}$$

Thus

$$\hat{\mathbb{E}}[\sup_{t \in [t_1, T]} |Y_t - Y_t^n|^\alpha] \leq \hat{\mathbb{E}}[(\frac{L_2}{n} + \frac{\|u\|_\infty}{n} |B_{t_1}|)^\alpha] \rightarrow 0.$$

By Proposition 3.8, we have

$$\hat{\mathbb{E}}[(\int_{t_1}^T |Z_s - Z_s^n|^2 ds)^{\alpha/2}] \leq C_\alpha \{ \hat{\mathbb{E}}[\sup_{t \in [t_1, T]} |Y_t - Y_t^n|^\alpha] + (\hat{\mathbb{E}}[\sup_{t \in [t_1, T]} |Y_t - Y_t^n|^\alpha])^{1/2} \},$$

where  $C_\alpha > 0$  is a constant depending only on  $\alpha$ ,  $T$ ,  $L^w$  and  $\underline{g}$ , thus we obtain  $\hat{\mathbb{E}}[(\int_{t_1}^T |Z_s - Z_s^n|^2 ds)^{\alpha/2}] \rightarrow 0$ , which implies that  $Z \in H_G^\alpha(t_1, T)$  for any  $\alpha > 1$ . By  $K_t = Y_t - Y_{t_1} + \int_{t_1}^t h(Y_s, Z_s) ds - \int_{t_1}^t Z_s dB_s$ , we obtain  $K_t \in L_G^\alpha(\Omega_t)$  for any  $\alpha > 1$ . We now proceed to prove that  $K$  is a  $G$ -martingale. Following the framework in Li and Peng [9], we take

$$h_i^n(x) = I_{[-n+\frac{i}{n}, -n+\frac{i+1}{n})}(x), i = 0, \dots, 2n^2 - 1,$$

$h_{2n^2}^n = 1 - \sum_{i=0}^{2n^2-1} h_i^n$ . Through equation (4.5), we get

$$\tilde{Y}_t^n = \tilde{Y}_T^n + \int_t^T h(\tilde{Y}_s^n, \tilde{Z}_s^n) ds - \int_t^T \tilde{Z}_s^n dB_s - (\tilde{K}_T^n - \tilde{K}_t^n),$$

where  $\tilde{Y}_t^n = \sum_{i=0}^{2n^2} u(t, -n+\frac{i}{n}, B_t - B_{t_1}) h_i^n(B_{t_1})$ ,  $\tilde{Z}_t^n = \sum_{i=0}^{2n^2} \partial_y u(t, -n+\frac{i}{n}, B_t - B_{t_1}) h_i^n(B_{t_1})$  and  $\tilde{K}_t^n = \sum_{i=0}^{2n^2} K_t^{-n+\frac{i}{n}} h_i^n(B_{t_1})$ . By Proposition 3.8, we have  $\hat{\mathbb{E}}[(\int_{t_1}^T |Z_s - \tilde{Z}_s^n|^2 ds)^{\alpha/2}] \rightarrow 0$  for any  $\alpha > 1$ . Thus we get  $\hat{\mathbb{E}}[|K_t - \tilde{K}_t^n|^\alpha] \rightarrow 0$  for any  $\alpha > 1$ . By Proposition 2.10, we obtain for each  $t_1 \leq t < s \leq T$ ,

$$\begin{aligned} \hat{\mathbb{E}}[|\hat{\mathbb{E}}_t[K_s] - K_t|] &= \hat{\mathbb{E}}[|\hat{\mathbb{E}}_t[K_s] - \hat{\mathbb{E}}_t[\tilde{K}_s^n] + \tilde{K}_t^n - K_t|] \\ &\leq \hat{\mathbb{E}}[|\hat{\mathbb{E}}_t[K_s - \tilde{K}_s^n|]|] + \hat{\mathbb{E}}[|\tilde{K}_t^n - K_t|] \\ &= \hat{\mathbb{E}}[|K_s - \tilde{K}_s^n|] + \hat{\mathbb{E}}[|\tilde{K}_t^n - K_t|] \rightarrow 0. \end{aligned}$$

Thus we get  $\hat{\mathbb{E}}_t[K_s] = K_t$ . For  $Y_{t_1} = u(t_1, B_{t_1}, 0)$ , we can use the same method as Part 1 on  $[0, t_1]$ .

Step 2.  $f(t, \omega, y, z) = \sum_{i=1}^N f^i h^i(y, z)$  with  $f^i \in M_G^0(0, T)$  and  $h^i \in C_0^\infty(\mathbb{R}^2)$ .

The analysis is similar to Part 2 of Step 1.

Step 3.  $f(t, \omega, y, z) = \sum_{i=1}^N f^i h^i(y, z)$  with  $f^i \in M_G^\beta(0, T)$  bounded and  $h^i \in C_0^\infty(\mathbb{R}^2)$ ,  $h^i \geq 0$  and  $\sum_{i=1}^N h^i \leq 1$ .

Choose  $f_n^i \in M_G^0(0, T)$  such that  $|f_n^i| \leq \|f^i\|_\infty$  and  $\sum_{i=1}^N \|f_n^i - f^i\|_{M_G^\beta} < 1/n$ . Set  $f_n = \sum_{i=1}^N f_n^i h^i(y, z)$ , which are uniformly Lipschitz. Let  $(Y^n, Z^n, K^n)$  be the solution of equation (3.1) with generator  $f_n$ .

Noting that

$$\hat{f}_s^{m,n} := |f_m(s, Y_s^n, Z_s^n) - f_n(s, Y_s^n, Z_s^n)| \leq \sum_{i=1}^N |f_n^i - f^i| + \sum_{i=1}^N |f_m^i - f^i| =: \hat{f}_n + \hat{f}_m,$$

we have, for any  $1 < \alpha < \beta$ ,

$$\hat{\mathbb{E}}_t[(\int_0^T \hat{f}_s^{m,n} ds)^\alpha] \leq \hat{\mathbb{E}}_t[(\int_0^T (|\hat{f}_n(s)| + |\hat{f}_m(s)|) ds)^\alpha].$$

Thus by Theorem 2.13, we get  $\|\int_0^T \hat{f}_s^{m,n} ds\|_{\alpha, \mathcal{E}} \rightarrow 0$  as  $m, n \rightarrow \infty$  for any  $\alpha \in (1, \beta)$ . By Proposition 3.9 we know that  $\{Y^n\}$  is a cauchy sequence under the norm  $\|\cdot\|_{S_G^\alpha}$ . By Proposition 3.7 and Proposition 3.8,  $\{Z^n\}$  is a cauchy sequence under the norm  $\|\cdot\|_{H_G^\alpha}$ . In order to show that  $\{K_T^n\}$  is a cauchy sequence under the norm  $\|\cdot\|_{L_G^\alpha}$ , it suffices to prove  $\{\int_0^T f_n(s, Y_s^n, Z_s^n) ds\}$  is a cauchy sequence under the norm  $\|\cdot\|_{L_G^\alpha}$ . In fact,

$$\begin{aligned} & |f_n(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| \\ & \leq |f_m(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| + |f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)| \\ & \leq L(|\hat{Y}_s| + |\hat{Z}_s|) + \hat{f}_n + \hat{f}_m, \end{aligned}$$

which implies the desired result.

Step 4.  $f$  is bounded, Lipschitz.  $|f(t, \omega, y, z)| \leq CI_{B(R)}(y, z)$  for some  $C, R > 0$ . Here  $B(R) = \{(y, z) | y^2 + z^2 \leq R^2\}$ .

For any  $n$ , by the partition of unity theorem, there exists  $\{h_n^i\}_{i=1}^{N_n}$  such that  $h_n^i \in C_0^\infty(\mathbb{R}^2)$ , the diameter of support  $\lambda(\text{supp}(h_n^i)) < 1/n$ ,  $0 \leq h_n^i \leq 1$ ,  $I_{B(R)} \leq \sum_{i=1}^{N_n} h_n^i \leq 1$ . Then  $f(t, \omega, y, z) = \sum_{i=1}^{N_n} f(t, \omega, y, z) h_n^i$ . Choose  $y_n^i, z_n^i$  such that  $h_n^i(y_n^i, z_n^i) > 0$ . Set  $f_n(t, \omega, y, z) = \sum_{i=1}^{N_n} f(t, \omega, y_n^i, z_n^i) h_n^i$ . Then

$$|f(t, \omega, y, z) - f_n(t, \omega, y, z)| \leq \sum_{i=1}^{N_n} |f(t, \omega, y, z) - f(t, \omega, y_n^i, z_n^i)| h_n^i \leq L/n$$

and

$$|f_n(t, \omega, y, z) - f_n(t, \omega, y', z')| \leq L(|y - y'| + |z - z'| + 2/n).$$

Noting that  $|f_m(s, Y_s^n, Z_s^n) - f_n(s, Y_s^n, Z_s^n)| \leq (L/n + L/m)$ , we have

$$\hat{\mathbb{E}}_t[\int_0^T (|f_m(s, Y_s^n, Z_s^n) - f_n(s, Y_s^n, Z_s^n)| + \frac{2L}{m}) ds]^\alpha \leq T^\alpha (\frac{L}{n} + \frac{3L}{m})^\alpha.$$

So by Proposition 3.9 we conclude that  $\{Y^n\}$  is a cauchy sequence under the norm  $\|\cdot\|_{S_G^\alpha}$ . Consequently,  $\{Z^n\}$  is a cauchy sequence under the norm  $\|\cdot\|_{H_G^\alpha}$  by Proposition 3.7 and Proposition 3.8. Now we shall prove  $\{\int_0^T f_n(s, Y_s^n, Z_s^n) ds\}$  is a cauchy sequence under the norm  $\|\cdot\|_{L_G^\alpha}$ . In fact,

$$\begin{aligned} & |f_n(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| \\ & \leq |f_m(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| + |f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)| \\ & \leq L(|\hat{Y}_s| + |\hat{Z}_s| + 2/m) + L/n + L/m, \end{aligned}$$

which implies the desired result.

Step 5.  $f$  is bounded, Lipschitz.

For any  $n \in \mathbb{N}$ , choose  $h^n \in C_0^\infty(\mathbb{R}^2)$  such that  $I_{B(n)} \leq h^n \leq I_{B(n+1)}$  and  $\{h^n\}$  are uniformly Lipschitz w.r.t.  $n$ . Set  $f_n = fh^n$ , which are uniformly Lipschitz. Noting that for  $m > n$

$$\begin{aligned} & |f_m(s, Y_s^n, Z_s^n) - f_n(s, Y_s^n, Z_s^n)| \\ & \leq |f(s, Y_s^n, Z_s^n)| I_{[|Y_s^n|^2 + |Z_s^n|^2 > n^2]} \\ & \leq \|f\|_\infty \frac{|Y_s^n| + |Z_s^n|}{n}, \end{aligned}$$

we have

$$\begin{aligned} & \hat{\mathbb{E}}_t[(\int_0^T |f_m(s, Y_s^n, Z_s^n) - f_n(s, Y_s^n, Z_s^n)| ds)^\alpha] \\ & \leq \frac{\|f\|_\infty^\alpha}{n^\alpha} \hat{\mathbb{E}}_t[(\int_0^T |Y_s^n| + |Z_s^n| ds)^\alpha] \\ & \leq \frac{\|f\|_\infty^\alpha}{n^\alpha} C(\alpha, T) \hat{\mathbb{E}}_t[\int_0^T |Y_s^n|^\alpha ds + (\int_0^T |Z_s^n|^2 ds)^{\alpha/2}], \end{aligned}$$

where  $C(\alpha, T) := 2^{\alpha-1}(T^{\alpha-1} + T^{\alpha/2})$ .

So by Theorem 2.13 and Proposition 3.7 we get  $\|\int_0^T \hat{f}_s^{m,n} ds\|_{\alpha, \mathcal{E}} \rightarrow 0$  as  $m, n \rightarrow \infty$  for any  $\alpha \in (1, \beta)$ . By Proposition 3.9, we conclude that  $\{Y^n\}$  is a cauchy sequence under the norm  $\|\cdot\|_{S_G^\alpha}$ . Consequently,  $\{Z^n\}$  is a cauchy sequence under the norm  $\|\cdot\|_{H_G^\alpha}$ . Now it suffices to prove  $\{\int_0^T f_n(s, Y_s^n, Z_s^n) ds\}$  is a cauchy sequence under the norm  $\|\cdot\|_{L_G^\alpha}$ . In fact,

$$\begin{aligned} & |f_n(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| \\ & \leq |f_m(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| + |f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)| \\ & \leq L(|\hat{Y}_s| + |\hat{Z}_s|) + |f(s, Y_s^n, Z_s^n)| I_{[|Y_s^n| + |Z_s^n| > n]}, \end{aligned}$$

which implies the desired result by Proposition 3.7.

Step 6. For the general  $f$ .

Set  $f_n = [f \vee (-n)] \wedge n$ , which are uniformly Lipschitz. Choose  $0 < \delta < \frac{\beta-\alpha}{\alpha} \wedge 1$ . Then  $\alpha < \alpha' = (1 + \delta)\alpha < \beta$ . Since for  $m > n$

$$|f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)| \leq |f(s, Y_s^n, Z_s^n)| I_{[|f(s, Y_s^n, Z_s^n)| > n]} \leq \frac{1}{n^\delta} |f(s, Y_s^n, Z_s^n)|^{1+\delta},$$

we have

$$\begin{aligned} & \hat{\mathbb{E}}_t[(\int_0^T |f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)| ds)^\alpha] \\ & \leq \frac{1}{n^{\alpha\delta}} \hat{\mathbb{E}}_t[(\int_0^T |f(s, Y_s^n, Z_s^n)|^{1+\delta} ds)^\alpha], \\ & \leq \frac{C(\alpha, T, L, \delta)}{n^{\alpha\delta}} \hat{\mathbb{E}}_t[\int_0^T |f(s, 0, 0)|^{\alpha'} ds + \int_0^T |Y_s^n|^{\alpha'} ds + (\int_0^T |Z_s^n|^2 ds)^{\frac{\alpha'}{2}}], \end{aligned}$$



where  $C(\alpha, T, L, \delta) := 3^{\alpha'-1}(T^{\alpha-1} + L^{\alpha'} T^{\frac{\alpha(1-\delta)}{2}} + T^{\alpha-1} L^{\alpha'})$ . So by Theorem 2.13 and Proposition 3.7 we get  $\|\int_0^T \hat{f}_s^{m,n} ds\|_{\alpha, \mathcal{E}} \rightarrow 0$  as  $m, n \rightarrow \infty$  for any  $\alpha \in (1, \beta)$ . By Proposition 3.9, we know that  $\{Y^n\}$  is a cauchy sequence under the norm  $\|\cdot\|_{S_G^\alpha}$ . And consequently  $\{Z^n\}$  is a cauchy sequence under the norm  $\|\cdot\|_{H_G^\alpha}$ . Now we prove  $\{\int_0^T f_n(s, Y_s^n, Z_s^n) ds\}$  is a cauchy sequence under the norm  $\|\cdot\|_{L_G^\alpha}$ . In fact,

$$\begin{aligned} & |f_n(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| \\ & \leq |f_m(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| + |f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)| \\ & \leq L(|\hat{Y}_s| + |\hat{Z}_s|) + \frac{3^\delta}{n^\delta}(|f_s^0|^{1+\delta} + |Y_s^n|^{1+\delta} + |Z_s^n|^{1+\delta}), \end{aligned}$$

which implies the desired result by Proposition 3.7.  $\square$

Moreover, we have the following result.

**Theorem 4.2** *Assume that  $\xi \in L_G^\beta(\Omega_T)$  for some  $\beta > 1$  and  $f, g$  satisfy (H1) and (H2). Then equation (1.2) has a unique solution  $(Y, Z, K)$ . Moreover, for any  $1 < \alpha < \beta$  we have  $Y \in S_G^\alpha(0, T)$ ,  $Z \in H_G^\alpha(0, T)$  and  $K_T \in L_G^\alpha(\Omega_T)$ .*

**Proof.** The proof is similar to that of Theorem 4.1.  $\square$

**Remark 4.3** *The above results still hold for the case  $d > 1$ .*

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